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# Spatial Econometrics Based on Kronecker Multiplication

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**Abstract**: **Objectives**: Countries and cities, as places of residence, are endowed with various dimensions, including economic and social ones. The interactions between these components manifest in flows that affect the quality of the human-made environment on one hand and the quality of human life on the other. **Prior Work**: One of the new methods for achieving higher accuracy in analyzing space-affected relationships is spatially weighted regression, introduced by Anselin and Griffith (1988). In his book "Spatial Econometrics: Methods and Models," he presented a comprehensive picture of spatial econometric realities for the first time, claiming this technique had better capabilities and applications than conventional econometrics in regional and spatial studies. **Approach**: The current research is mainly done by a review method. **Results**: The result of this article is the introduction of spatial econometrics based on Kronecker's multiplication, which is used in many researches, especially researches related to sustainable development. **Implications**: This article aims to familiarize the reader with spatial econometrics and some features of the Kronecker product, as well as some of its applications in the context of spatial dependency, spatial heterogeneity, the nature of adjacency in spatial econometrics, spatial positioning, spatial lags, autoregressive models, mixed regression models, spatial autoregression, and the Kronecker product, discussed briefly in this paper. **Value**: As far as we know, this article is one of the first articles that introduced, described and explained the spatial econometric method based on Kronecker's multiplication, so it is innovative.

**Keywords**: Spatial econometrics; Kronecker Multiplication; adjacency; spatial dependency; spatial heterogeneity

**JEL Classification**: C21; C1

## 1. Introduction

In recent years, a mounting global challenge has emerged in the form of climate change and environmental governance (Zhu, Su, Fan, Qin & Fu, 2024). To doing good researches in this regards, using suitable methods is very important. Also the use of techniques, methods, and quantitative tools in various sciences has a long history and has expanded significantly today. Sciences have utilized quantitative and numerical methods based on their degree of objectivity. Among these, the field of economics has also been subject to methodological debates. Keynes believed that the growth and expansion of economics in the style and method of physics were not possible, stating in a letter to

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Harrod: "Economics is a normative science, not a natural science, and in this science, introspection and value judgments should be used more". Keynes argued that converting economic models into quantitative formulas would destroy their usefulness as tools for thought because economics is a method and tool for the mind, a technique for thinking that gives the thinker the ability to extract results.

The criticisms and challenges created among scholars for and against the use of quantitative methods in economics, rather than halting the spread of quantitative methods in economics, led to the refinement of techniques and made experts aware of the shortcomings of quantitative methods in normative and behavioral sciences. This gave rise to a branch of economics called econometrics, which has engaged many enthusiasts over the past few decades and has made significant progress in various aspects of nature and method. From the first bibliography written on the method of least squares, published in 1877 at Oxford University, to today, where various econometric techniques are used for estimation, assessment, and forecasting (Asghari et al., 2001).

Nowadays, many scientific studies require the use of numerical and statistical information influenced by the concept of space and environment. In such studies, the extent and manner of space's impact are important, and ignoring the effect of space leads to errors in estimation and forecasting. Manuel Castells, a professor of urban and social planning at the University of California, defines and interprets geographical space as follows: "Space is the material production in relation to other material factors." Among other factors, humans are situated within social relationships, especially giving space its form and social function. The concept of geographical space is particularly suitable for analyzing the human environment (Soltani et al., 2014). This article aims to introduce and examine the domain and various applications of spatial econometrics in regional sciences.

## 2. Theoretical Framework

Space is a subject discussed in sciences such as mathematics, astronomy, physics, chemistry, economics, accounting, sociology, architecture, urban planning, and geography, and may be defined differently in each of these fields (Afrough, 1998).

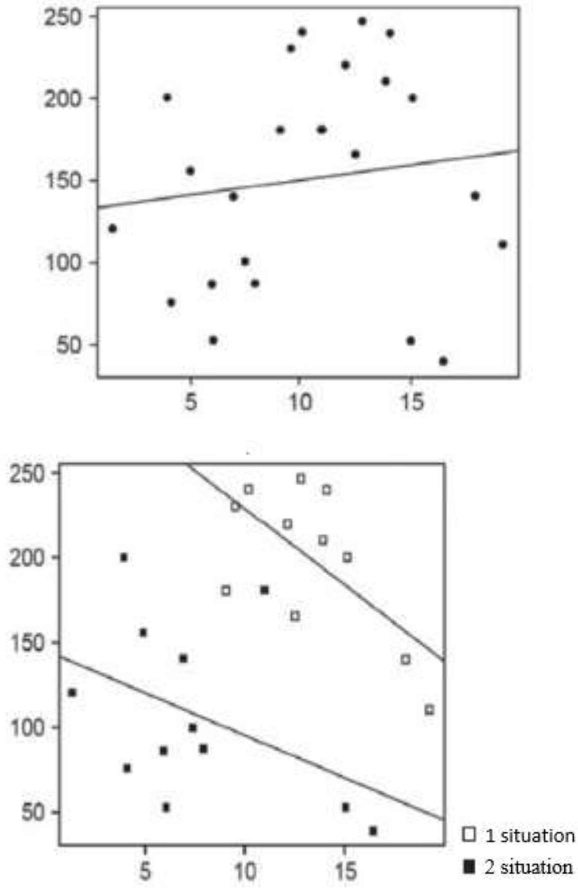
In spatial statistics, we usually deal with data that have spatial aspects. Therefore, before anything else, it is necessary to determine the quantity and numerical value of the spatial aspect. To accomplish this, two sources of information are available: one is the position on the coordinate plane, which is expressed through geographical longitude and latitude, and based on this, it is possible to calculate the distance of any point in space or the distance of any observation located at any point relative to fixed or central points or observations. Therefore, observations that are closer to each other reflect a higher degree of spatial dependency compared to those that are farther apart. The second source of spatial information is proximity and adjacency, which reflects the relative position in the space of an observed regional unit in relation to other such units. Based on this information, it can be determined which areas are neighbors or adjacent, meaning they have borders that meet. Units that have a correlation or adjacency relationship show a higher degree of spatial dependency compared to locations or units that are farther away (Akbari, 2001).

An important point here is the general estimation of the relationship, which sometimes leads to misleading and erroneous interpretations of local relationships.

The image below is very important in understanding this matter. In this image, a spatial example of this type of contradiction, known as Simpson's paradox, is observed. Figure 1 show an example of

Simpson's paradox in case A, with undifferentiated spatial data, and in case B, with differentiated spatial data, when the regression line passes through the data (Brunsdon et al., 2002).

A




B

**Figure 1. An Illustration of Simpson's Paradox**

*Source: Brunsdon et al., 2002*

Simpson's paradox generally refers to the reversal of results. When groups of data are examined separately, regardless of spatial coordinates, and when combined, we face two behaviors. In the example shown in the figure above, the plotted data show the relationship between house prices and the population density of the area where the house is located.

In chart (A), the data from random observations indicate that there is a direct linear correlation between population density and house prices. While in chart (B), the data are differentiated based on spatial location, and in two positions, the linear relationship between house prices and population density is negative. Simpson's paradox, in fact, expresses the role of the spatial position variable in explaining the relationships between variables and indicates the important point that ignoring the space factor may lead to incorrect interpretations of the spatial relationships of variables. In general, the difference between spatial statistics (affected by the space factor) and general statistics (without considering spatial position) can be expressed as Table 1, which emphasizes the positive features of spatial statistics (Brunsdon et al., 2002).

**Table 1. The Difference between Spatial Statistics and General Statistics without Considering Spatial Position**

| | Spatial | General |
|---|---|---|
| Ability to address details | Local details from quantitative information | Summarizing information for the entire region level |
| Dimensions of analysis | Multivariate | Univariate and Multivariate |
| Ability to express and illustrate | Better ability to illustrate and express graphically | General illustration |
| Compatibility with GIS | Compatible with GIS and has coordinate justification | Incompatible with GIS and lacks coordinate justification |
| Analytical approach | Emphasis on differences across space | Emphasis on similarities across space |
| Analytical method | Spatially weighted regression | Ordinary regression |

*Source: Brunsdon et al., 2002*

When the relationship between independent and dependent variables becomes negative in one part of the study domain and positive in another, the ordinary regression model will not be able to accurately detect the relationship between the independent and dependent variables.

While the traditional econometric model is a widely adopted method for influencing factor analysis, it disregards the impact of spatial correlation on provincial carbon-pollutant emissions efficiency (CPEE) for example (Zhu et al., 2024).

In addition, the analysis of spatial data requires different approaches to the relationship between independent and dependent variables in the model (Soltani et al., 2014).

## 3. Spatial Econometrics

Research work in regional sciences is widely based on regional sample data. The distinction between spatial econometrics and conventional econometrics lies in the ability of spatial econometrics to use sample data that have a spatial component. The presence of a spatial component in sample data creates two problems ((Lesage, 1999, p. 1) spatial dependency between observations and 2) spatial heterogeneity, which conventional econometrics largely ignores, perhaps because they disregard the Gauss-Markov assumptions used in the regression model.

The existence of spatial dependency among samples violates the Gauss-Markov assumption that explanatory variables in repeated samplings are constant. Also, the Gauss-Markov theorem assumes that there is a specific hypothetical relationship between sample observations.

The presence of spatial heterogeneity among samples violates the assumption because the relationship between sample observations will change with the movement among spatial sample data due to spatial dependency. In this case, the coefficients will not be a linear function of the dependent variable. Therefore, spatial econometric methods must be employed.

According to the Gauss-Markov theorem, sample regression data can be considered as the following relationship 1:

$$Y=X\beta + \varepsilon \quad (1)$$

Where Y represents a vector of n observations, X is an n times k matrix of explanatory variables, beta is a vector of k parameters, and epsilon is a vector of n random error terms. The data generation process is such that the matrix X and the correct parameters beta are constant, and as a result, the distribution of the sample vectors Y has a covariance structure similar to epsilon. According to the Gauss-Markov theorem, the distribution of observations in Y is such that when moving among observations, a constant value will be shown, and as a result, the covariance between observations is zero; while in sample data that have spatial dependency and spatial heterogeneity, this phenomenon will not exist (Asghari et al., 2001).

In regional sciences, we face data that have spatial properties. In geolocated data, the Gauss-Markov assumptions are violated due to the creation of spatial dependency (spatial autocorrelation) and spatial heterogeneity (spatial variations) among observations. Therefore, for accurate modeling, a method other than conventional econometrics is needed, which is expressed in spatial econometrics. The simple reason for using spatial econometrics is to follow the first law of geography, which says that everything (such as labor force, capital force, etc.) is related to other things, but things that are closer to each other are more related than those that are farther apart.

In the ordinary linear regression model, spatial data in all studied regions are assumed to be static. In other words, it can be written as 2:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_n X_{ni} + \varepsilon_i \quad (2)$$

$i = 1, 2, \ldots, n$

The parameter estimates obtained from this model are constant. Therefore, we will have 3:

$$\hat{\beta} = (X^T X)^{-1} X^T Y \quad (3)$$

According to the Gauss-Markov assumptions, it is assumed that the explanatory variables in repeated samplings are constant.

The existence of spatial dependency among samples violates this assumption. Also, spatial heterogeneity violates the Gauss-Markov assumption that there is a specific linear relationship between sample observations. Because with the assumption of spatial dependency among data, the relationship will change with movement between spatial sample data, and the coefficients will not be a linear function based on the dependent variable.

The issue of spatial heterogeneity (spatial variations) of data and the analysis of spatial variations in relationships can be directly presented by the method of geographic weighted regression. The regression model in the geographic weighted regression method can be considered as relationship 4:

$$Y_i = \beta_0(i) + \beta_1(i) X_{1i} + \beta_2(i) X_{2i} + \cdots + \beta_n(i) X_{ni} + \varepsilon_i \quad (4)$$

The estimation of coefficients in this method will be as follows:

$$\hat{\beta}(i) = (X^T W(i) X)^{-1} X^T W(i) Y$$

Where W(i) is the weight matrix based on position i (based on geographical longitude and latitude), such that observations closer to i have greater weights compared to observations farther from i. This matrix can be defined as follows:

$$W(i) = diag[w_{i1}, w_{i2}, \ldots, w_{in}]$$

Or:

$$W(i) = \begin{bmatrix} W_{i1} & \cdots & \cdots & 0 \\ 0 & W_{i2} & . & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & . & . & W_{in} \end{bmatrix}$$

Where Win is the weight given to data point n for estimating regional parameters at position i. Equation (4) can also be written as relationship 5:

$$Y_i = \beta_0(u_i, v_i) + \beta_1(u_i, v_i)X_{1i} + \beta_2(u_i, v_i)X_{2i} + ... + \beta_n(u_i, v_i)X_{ni} + \varepsilon_i \quad (5)$$

Where ui and vi are the geographical longitude and latitude, respectively.

Unlike ordinary regression models, the estimation of coefficients of explanatory variables in the geographic weighted regression model is not constant and provides a unique coefficient for each location (Samadi et al., 2015).

## 4. How to Determine Location in Spatial Econometric Models

To determine location in spatial econometric models and form adjacency matrices, there are two methods:

A) Determining Spatial Position

The adjacency matrix can be defined based on the element of distance. Observations that are closer to each other should reflect a higher degree of spatial dependency compared to those that are farther apart. In other words, spatial dependency and its effects among observations should decrease with increasing distance. Therefore, this matrix is formed based on the inverse of the distance between each observation and other observations or the inverse of the square of the distance. Accordingly, the elements on the main diagonal of the adjacency matrix are zero, and the other elements of the matrix represent the inverse of the distance between each observation and others.

In the case of forming the adjacency matrix, the elements on the main diagonal are zero, and the other elements are based on whether the countries are adjacent or not, taking the number one or zero. In spatial econometric models, it is assumed that each spatial section is not its own neighbor. Not following this assumption leads to results that are significantly complex and not easily interpretable. Spatial spillover does not only affect one neighbor of the study area but also affects the neighbors of that neighbor, and this chain continues until the spillover effects reach the boundary of the study area. First-order neighbors are the closest neighbors to the spatial section in question. Second-order neighbors are the neighbors of the first-order neighbors. Third-order neighbors are the neighbors of the second-order neighbors. Subsequently, the adjacency matrix must be standardized, which is called the standardized first-order adjacency matrix. By standardizing the adjacency matrix and then multiplying it by the dependent variable vector, a new variable is obtained that shows the average of observations from adjacent areas, commonly referred to as the spatial lag variable (Ziari and Shafiei-Kakhki, 2016: 51).

B) Determining Spatial Adjacency

Using adjacency and neighborhood reflects the relative position in space of one regional unit observed compared to other similar units. In other words, by determining which areas are neighbors or adjacent, the adjacency matrix is formed. Therefore, considering spatial dependency, units that have a neighborly or adjacent relationship should show a higher degree of dependency compared to places that are farther apart. There are various methods for forming the adjacency matrix, including linear adjacency, face-to-face adjacency, elephant-like adjacency, two-sided linear adjacency, two-sided face-to-face adjacency, and queen-like adjacency (Ziari & Shafiei-Kakhki, 2016).

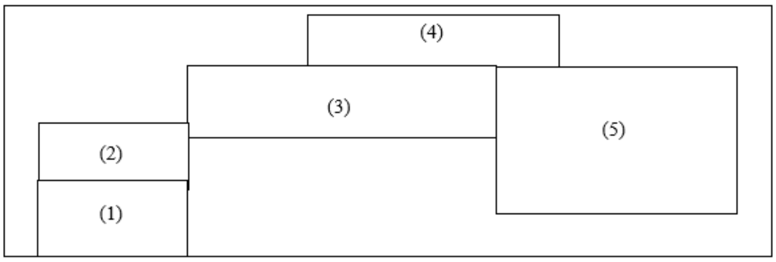



**Figure 2. Shows the Adjacency between Five Regions (Askari et al., 2001)**

To determine adjacency, there are different methods, which will be explained below, some of the different methods of defining the W square matrix that represents different definitions of “adjacent” relations among the five regions shown in Figure 2.

**Linear adjacency**: Wij = 1 is defined for elements that have an immediate shared edge with the right or left of the region under study. For row 1, which reflects the connections related to region one, there is no immediate right or left adjacency. On the other hand, for row 5, which records the connection related to area 5, W53 = 1, and the other row elements will be equivalent to zero.

**Face-to-face adjacency**: Wij = 1 is defined for regions that have a common side with the region under study. For row 1, reflecting the connections of region 1, we have W53 = 1, and the other elements of this row will be equivalent to zero. As another example, in row 3, we have W34 = 1 and W35 = 1, and the rest of the row elements are zero.

**Elephant-like adjacency**: Wij = 1 is defined for elements that have a common vertex with the region under study. For region (2), we have W23 = 1 and the other row elements will be zero.

**Two-sided linear adjacency**: Wij = 1 is defined for two regions that are immediately to the right or left of the region under study. This definition for the regions shown in Figure 1 produces the same results as linear adjacency.

**Queen-like adjacency**: Wij = 1 is defined for existing regions that have a common side or vertex with the region under study. For region 3, we will have W32 = 1, W34 = 1, W35 = 1, and the other elements are zero.

Sometimes the definitions of two-sided linear and two-sided face-to-face are referred to as “second-order” adjacency, while for other definitions, the term “first-order” is used.

Sometimes the definitions rely more on the details, focusing on the amount of distance of shared borders. This raises the question of whether regions (4) and (5) in Figure 2 are adjacent or no, Because the two regions share a common border, but that border is very short.

The main reason for choosing a definition of adjacency should be related to the nature of the problem that is to be modeled. For example, suppose there is a major highway connecting regions (2) and (3), and region (2) is a “dormitory complex” for people working in region (3). With this information known, an adjacency model should be selected that shows a high spatial reaction between these two regions. Or suppose the areas in question are only adjacent to the left or right and are able to influence their neighboring areas, but not the upstream and downstream areas. In this case, a linear type of adjacency will necessarily be used.

The W matrix, which reflects the first-order contiguity relationships for the five regions shown in Figure 2, is as follows:

$$W(i) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Usually, the definition of contiguity is used in applied studies, the main reason being that the contiguity definition encompasses all regions that share a common border.

Note that the matrix, as shown, is different. And according to convention, the matrix always has zero elements on the main diagonal. A transformation that is often used in applied works is to convert the matrix into a matrix whose row sum is one. This is referred to as the "standardized first-order contiguity matrix," which we denote as matrix C (Asghari et al., 2001).

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

## 5. Spatial Lags

One of the fundamental concepts related to spatial adjacency is spatial lag. Spatial lags are similar to lagging in time series analysis, where Byt = yt-1 represents the first-order lag and BPyt = yt-p indicates the pth order lag. Unlike the time domain, spatial lag refers to a shift across space, but it is limited by certain constraints that arise when one tries to draw parallels between the domains of time and space.

In studies with spatially dimensional data, the concept of spatial lag means observations that are one or more distance units away from a specific location, where distance units can be measured in two or four directions. In practical situations, observations may not represent a regular network or sequence, as they are irregularly plotted on the map of regions and the concept of spatial lag relates to a set of neighbors pertaining to a specific location. In this concept, the operation of spatial lag acts to create a weighted average of neighboring observations.

As mentioned, the concept of "neighbors" in spatial analysis is not fixed but depends on the definition used. Like time series analysis, it seems logical to increase the order of the first-order square adjacency matrix W, which contains values of 0 and 1; hypothetically equivalent to p, to create a spatial lag.

In spatial econometrics, we face a process where spatial spillover effects operate over time. Over time, the initial effects on neighbors influence more and more regions. The spillover effect should logically be considered for the outward flow from neighbor to neighbor, and the concept of spatial lag encompasses this idea (Ziari & Shafiei-Kakhki, 2016).

## 6. Spatial Autoregressive Models

Anselin (1988) introduced several spatial autoregressive models applicable to cross-sectional spatial data as follows:

$$Y = pw1Y + X\beta + U$$

$$U = \lambda W2U + \varepsilon$$

$$\varepsilon \sim N(0, \sigma 2\ ln)$$

Where Y consists of n times 1 vector of dependent cross-sectional variables and X represents an n times k matrix of explanatory variables.

W1 and W2 are n times n spatial weighting matrices (based on Kronecker product, which will be discussed later) that usually include first-order adjacency relations or functions of distance. A first-order adjacency matrix has zero elements on the main diagonal, meaning rows containing zero elements relate to non-adjacent observational units, and one element reflect neighboring units that are adjacent based on one of the adjacency definitions (Ziari and Shafiei-Kakhki, 2016).

Using the above model, spatial models can be derived by applying certain constraints.

For example, if:

X = 0 and W2 = 0, a first-order spatial autoregressive model is created as follows.

$$Y = pw1Y + \varepsilon$$

$$\varepsilon \sim N(0, \sigma 2\ ln)$$

This model is called a first-order autoregressive model because the primary reliance is on past period observations to explain deviations in Y.

Assuming W2 = 0, a mixed spatial regression-autoregressive model is created (as described below), where we have additional explanatory variables in the matrix X used to explain deviations in Y across the spatial sample of observations.

$$Y = pw1Y + X\beta + \varepsilon$$

$$\varepsilon \sim N(0, \sigma 2\ ln)$$

Assuming W1 = 0, a regression model with spatial autocorrelation (general form) is shown as follows.

$$Y = X\beta + U$$

$$U = \lambda W2U + \varepsilon$$

$$\varepsilon \sim N(0, \sigma 2\ ln)$$

This section is divided into subsections, each examining and demonstrating the spatial autoregressive model like the general form (to be discussed later).

### 6.1. Kronecker Product

There are various methods for multiplying two matrices, including ordinary multiplication, Hadamard product, and Kronecker product. The Kronecker product, denoted by $\otimes$, was first used by Johann Georg Zehfuss in 1858 and is also known as direct product or tensor product.

Many applications lead to the production of matrices with high dimensions. One of the methods used to save memory is to decompose a high-dimensional matrix into lower dimensions. For example, in Figure 3 on the left, a matrix of dimensions 9×9 is shown, which is transformed into two matrices of dimensions 3×3 using the Kronecker product, thus requiring less space for storage (Afshari & Golpar, 2018).

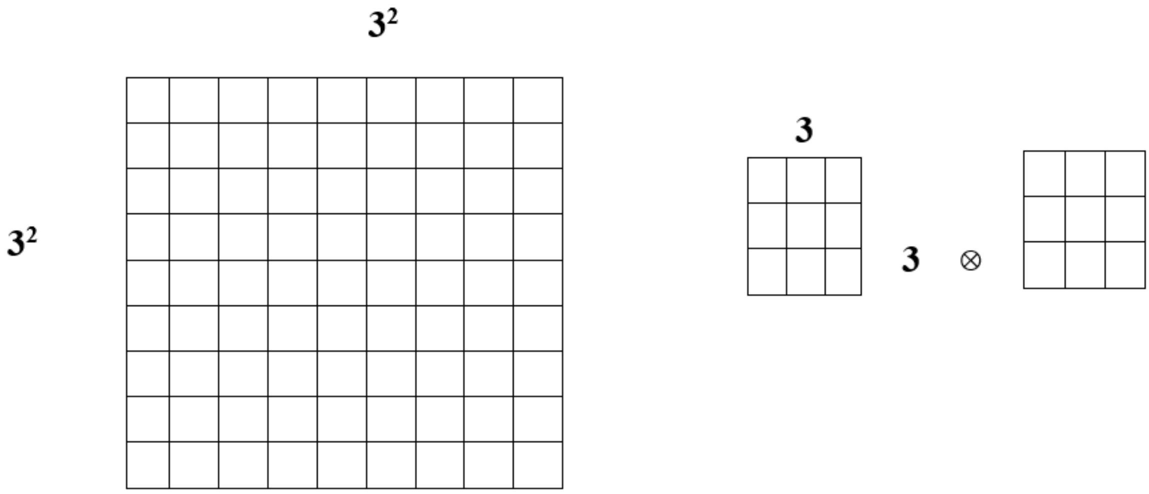



**Figure 3. The Use of the Kronecker Product in Compression (Afshari & Golpar, 2018)**

**Vectorization operation:** Suppose $A = (a_1 ... a_n) \in C^{m\times n}$ ai is the ith column of matrix A. The vectorization operator text vec converts matrix A into an mn dimensional vector obtained by stacking the columns of A. In other words:

$$vec(A) = \begin{pmatrix} a_i \\ . \\ . \\ . \\ a_n \end{pmatrix}$$

Theorem 1. Suppose $A \in C^{pr\times qs}$, then $A = B \otimes C$ if and only if $rank\lfloor vec(A_{11}),...,vec(A_{pq})\rfloor$ A is as follows:

$$A = \begin{bmatrix} A_{11} & . & . & . & A_{1q} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_{P1} & . & . & . & A_{pq} \end{bmatrix}$$

$A$And every $A_{ij} \in C^{r\times s}$ that is $1 \le i \le P$ and $1 \le j \le q$.

If matrix A cannot be decomposed into the Kronecker product of two matrices, matrices B and C are determined in such a way that they have the least distance to the original matrix. In other words, $\|A - (B \otimes C\| F$ for $B \in C^{p\times q}$ and $c \in C^{r\times s}$ has the least value (Afshari Arjmand and Golpar Rabooki, 2018).

### 6.2. Applications

### 6.2.1. Transformations

Transformations such as Fourier, cosine, and wavelet can be represented in a compact form using the Kronecker product. In this way, the matrix corresponding to higher-order transformations can be calculated using the Kronecker product and a lower-order transformation matrix. For example, we examine the Haar and Hadamard wavelet transformations.

The matrix corresponding to the Haar wavelet can be expressed using the Kronecker product as follows:

$$W_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, W_n = \begin{pmatrix} W_{n-1} \otimes (1 \quad 1) \\ I_2 \otimes I_{2^{n-2}} \otimes (1 \quad -1) \end{pmatrix}$$

which shows that the wavelet matrix of order \( n \) is constructed using the Kronecker product and the wavelet matrix of stage \( n-1 \). The Hadamard transformation can also be defined as follows:

$$H_1 = (1), H_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, H_{2^k} = \begin{pmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{pmatrix} = H_2 \otimes H_{2^{k-1}}, 2 \leq k \in N$$

Which indicates that the Hadamard transformation at each stage is constructed using the Kronecker product of the previous stage's Hadamard transformation and H2 (Stanković, 2003).

### 6.2.2. Generating Fractals by a Primitive Pattern

A fractal is a geometric structure that, when magnified by a certain ratio, reproduces the original structure; this property is called self-similarity.

Given the mentioned properties for a fractal and having its primitive pattern, one can reach the complete shape of the fractal using the Kronecker product. Suppose matrix M is the primitive pattern of the fractal; the following program in MATLAB environment generates and displays the corresponding fractal (Stiep, 2008):

$N = Kron(M, M)$

**For i=1:k**

$N = Kron(M, N)$

**end:**

**Imshow(N):**

For matrices:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, M_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Similar fractals consist of:

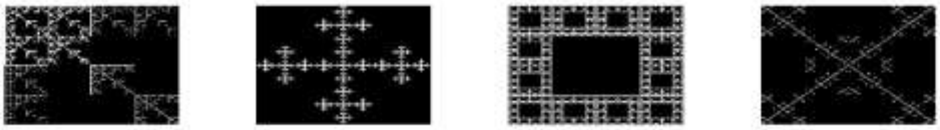


**Figure 4. Corresponding fractals M1, … M4 from left to right (Stiep, 2008)**

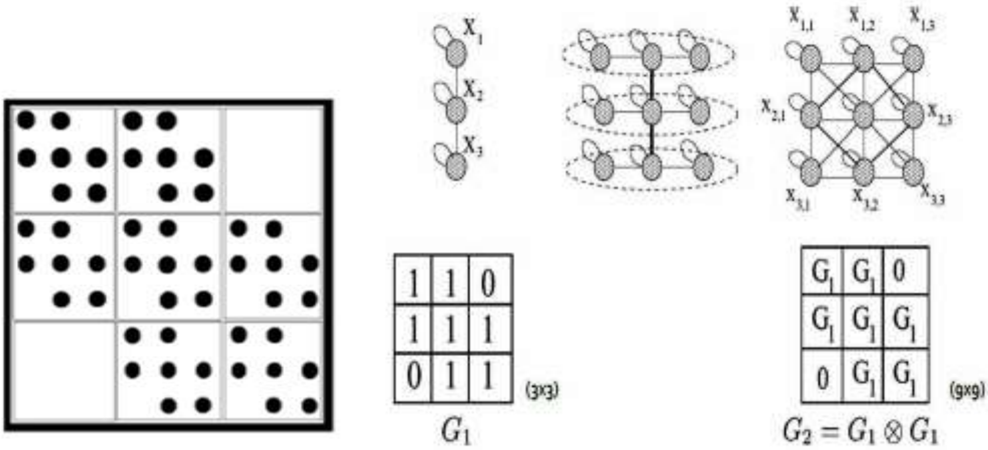
### 6.2.3. Kronecker Graphs

Each graph G is associated with a square matrix called the adjacency matrix, the size of which is equal to the number of vertices of the graph, such that the ijth entry is equal to the number of edges connecting vertex i to vertex j.

A Kronecker graph of order n with adjacency matrix is a graph obtained from $G_n = \underbrace{G_1 \otimes G_1 \otimes ... G_1}_{n \text{ بار}} n$

the Kronecker product of the initial adjacency matrix G1.

The self-similar nature of the Kronecker product is evident. To generate Gi from Gi-1, we replace each group of Gi-1 with the initial adjacency matrix G1, as indicated in Figure 5 (Afshari & Golpar, 2018).




A B

**Figure 5. (a) Adjacency Matrix $G_2$ of the Related 9×9 Graph and (b) the Corresponding Fractal of the Graph's Adjacency Matrix (Afshari & Golpar, 2018)**

### 6.3. Solving Matrix Equation Systems

Using the Kronecker product, a system of matrix equations containing a matrix of unknowns (X) can be transformed into a system of linear equations.

Among the most important matrix equation systems and their corresponding linear systems, the following can be mentioned (Afshari & Golpar, 2018):

$$AX = B \Rightarrow (I \otimes A)vec(X) = vec(B),$$
$$AX + XB = C \Leftrightarrow [(I \otimes A) + (B^t \otimes I)]vec(X) = vec(C),$$
$$AXB = C \Leftrightarrow (B^t \otimes A)vec(X) = vec(C),$$
$$AX + YB = C \Leftrightarrow (I \otimes A)vec(X) + (B^t \otimes I)vec(y) = vec(C).$$

## 7. Spatial Autoregressive Models

Anselin and Griffith introduced several spatial autoregressive models applicable to cross-sectional spatial data as follows:

$$Y = PW_1Y + X\beta + U$$
$$U = \lambda W_2U + \varepsilon$$
$$\varepsilon \approx N(0, \sigma^2 In)$$

Where Y includes an n times 1 vector of dependent cross-sectional variables, and X represents an n×k matrix of explanatory variables. $W_1$ and $W_2$ are n×n spatial weighting matrices that usually include first-order adjacency relations or functions of distance. Using the general model above, spatial models can be derived by applying certain constraints (Ziari & Shafiei-Kakhki, 2016).

### 7.1. Spatial Error Model (SEM)

Among other models proposed in the field of spatial econometrics is the spatial error model. Anselin and Griffith (1988) used this model as follows

$$Y = x\beta + u$$
$$u = \lambda wu + \varepsilon$$

Where Y includes an n×1 vector of dependent variables, and x denotes an ordinary n×k statistical matrix containing explanatory variables. W is known as the spatial weighting matrix, and the parameter lambda is the coefficient of spatially correlated errors, similar to the issue of component-wise correlation in time series models. The parameter beta indicates the effect of explanatory variables on the deviation in the dependent variable y.

The original model proposed by Anselin (1988) for cross-sectional characteristics was called the Spatial Autoregressive (SAR) and Spatial Error Model (SEM). This could be the result of any of those econometric models that start with an economic stimulus. A stimulus for the spatial regression model arises from observing spatial dependence as a long-term equilibrium of a spatio-temporal process; and a stimulus for the spatial error model demonstrates that omitted variables that precede spatial dependence to a spatial lag model where both explanatory and dependent variables are represented.

The spatial regression model includes a spatial lag of the dependent variable. Our second econometric driver leads us to a model with a spatial lag where both explanatory and dependent variables are present. However, if the inclusion and exclusion (deprivation) of the explanatory variable are not related, a spatial error model appears. (Ziari & Shafiei-Kakhki, 2015).

### 7.2. Spatial Durbin Model (SDM)

In the methodology of spatial econometrics, depending on whether the dependent variable, explanatory variables, or the error term have spatial dependence, different spatial models are proposed. The Spatial Durbin Model (SDM) holds a special place among spatial models. A distinctive feature of this model compared to other spatial models (such as SAR and SEM) is the simultaneous introduction of spatial lags of the dependent variable and spatial lags of the explanatory variables as new explanatory variables in the model. (Ziari & Shafiei-Kakhki, 2015).

## 8. Conclusion

Nowadays, many scientific studies contain statistical information where proximity and distance play a significant role. Therefore, spatial econometrics techniques can be effectively used among various scholars in fields such as economics, sociology, geography, international economics, urban and regional economics, etc., and provide interesting results. The results of this research are summarized as follows:

1. Spatially collected sample data necessarily do not have a constant mean and variance, and in this case, the Gauss-Markov assumptions are not reliable, and relying on the Gauss-Markov assumptions is not possible when using such data.

2. All conventional econometric models have the capability to be adjusted and transformed into spatial econometric models when employed with spatial data.

3. If spatial econometrics is used instead of conventional econometrics, all the capabilities of conventional econometrics are preserved, in addition to some of the estimates being more accurate.

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