Modern Paradigm Regarding Capital Markets: Fractal Market Hypothesis. Determination of the Hurst Exponent on the Romanian Capital Market

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Abstract: Classical statistical and econometric theory, intended to provide functional forecasting models in capital markets, is the mathematical foundation for a number of theories - efficient market theory, Harry Markowitz's optimized portfolio theory, the CAPM model developed by Sharpe, and the modern theory of portfolio - Modern Portfolio Theory (MPT). Although capital markets are considered at their best to be efficient, in real world, testing their efficiency proved eroneous. As the concept applies normally to random processes, or Brownian motions. But in very rare moments markets are efficient (weak or strong form of efficieny), rather than they follow fractional Brownian motions. In other words, for this particular type of process, the memory effect over the price is encompassed. For determining which cathegory of process the price may follow, R/S Analysis is a robust tool for testing whether a historical price series, for certain time interval, follows a Brownian motion or there is some memory effect over it. Determining the Hurst exponent for company ALRO S.A., for a the period of time since listing, until 16/07/2021 was the aim of this paper. Values of the Hurst Exponenent indicate important information regarding memory effect inside the stock market, or appearance of some random process. As a conclusion, Romanian capital market has evolved from a very low stability market to a more stable investment environment, meaning increasing liquidity over the market.

Keywords: efficient market hypothesis; fractal market hypothesis; capital market; R/S analysis; Hurst Exponent; capital market

1. Introduction

The purpose of the paper was to present theoretical implications of the FMH and to apply R/S analysis on the historical time series of company Alro S.A. The results of the analysis indicate translation from an antipersistent trend, going to a random walk phase and reaching a persistent trend phase. The Hurst Exponent indicate the presence of a long-term memory effect, in the case H is different from 0.5. As the value of this metric is 0.5, it detects the independence of the studied series, but offers no indications about distribution.

Classical statistical and econometric theory, intended to provide functional forecasting models in capital markets, is the mathematical foundation for a number of theories - efficient market theory, Harry Markowitz's optimized portfolio theory, the CAPM model developed by Sharpe, and the modern theory of portfolio - Modern Portfolio Theory (MPT).

There are suggestions among capital market theorists that efficient market hypothesis has only one function, which is to justify the use of probabilistic calculus in the analysis of capital markets. Specifically, the fame of this theory is demonstrated by thousands of studies and tests performed on it, being one of the most controversial theories, due to the ambiguity and general distrust under which the

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results are - confirmation of the efficient market hypothesis, with a certain statistical probability, it does not necessarily mean the efficiency of the capital market, but the fact that it tends to become efficient (and subsequently may result in the use of predictability models may be useless), and the rejection of the efficient market hypothesis does not necessarily mean that the market is inefficient.

From mathematical perspective, the very first formulation for EMH was the random walk version, being also the most restrictive version. Market efficiency does not necessarily imply a random walk, but a random walk implies for sure market efficiency. Alternatively, the independence assumption between market moves conducted to a more general martingale or submartingale models for market efficient hypothesis.

One of the theories that fundamentally marked the modeling of financial phenomena is the efficient market hypothesis (Fama, 1970), according to which in an efficient market prices always correctly and completely reflect the available information. A particular form of efficiency, weak form efficiency, has the effect that trading prices follow a random walk pattern and returns are unpredictable.

Guerrien and Gun (2011) take an extremely critical position against the efficient market hypothesis, showing that the idea of an efficient market cannot be valid from the point of view of Pareto optimality, as the conditions to be met in this regard are too restrictive.

Another widely spread idea among specialists is that in an efficient market it would not be possible to form speculative bubbles, or the formation of a speculative bubble market regime (Abreu & Brunnermeier, 2003), or they are quite common in recent history of capital markets.

Malkiel (2011) refines the discussion on the relationship between the efficient market hypothesis and the financial crisis, dismantling much of the criticism of this hypothesis.

Malkiel highlights the two main implications of the hypothesis: the fact that public information is reflected in price without delay, and the lack of opportunities for arbitrage. The author points out that in an efficient market prices are not always „correct” and not all investors necessarily behave rationally. Moreover, the fact that the trading price is “fair” refers to the fact that, according to the efficient market assumption, it is impossible to assess whether the price is undervalued or overvalued at any given time, which is an area dominated by uncertainty.

2. Fractal Market Hypothesis vs. Efficient Market Hypothesis

Fractal market hypothesis (FMH) was developed by Peters (1989, 1992) as a replica of efficient market hypothesis. According to FMH, the emphasis is no longer on market efficiency, but on its stability. The market is considered stable when it is “liquid”, in the sense of an unbalanced trading volume. Also, unbalanced trading volume represents a simplifying metaphor for an effective distribution among demand and supply, in other words, among “bulls” and “bears” in the market. If there is enough dis-equilibrum (disharmony) between those two forces applying on the price, the stability of the market fades and markets become turbulent. If a market is liquid, then the market price is close to the “fair” one, the volume of transactions being large enough so that the market can find the full range of possible expectations of investors regarding the trading price, and it has is formed as a result of the normal balance between supply and demand.

Contrary to the theory of efficient markets, fractal market theory considers that information does not have a uniform impact on prices, information is assimilated differently, depending on different investment horizons. Different time horizons evaluate information differently. At any given time, the prices may not reflect all the information available, but only the information that is important for the investment horizon.

As a matter of fact, it is not information, but investment horizons of different types of investors are the crucial factor that determine liquidity, in other words, stability in the markets. Due to different approaches, a daily trader quantifies different information he receives regarding his daily trading strategy from an investment fund. For him, the impact of a financial news may lead to a “sell” order regarding a stock in his portfolio, at the end of the day, as for a trade fund manager may see as an opportunity to “buy” that same stock, generating an opposite position, and, at the end of the same day, the investment fund successfully “buys” and “closes” open positions of tens of daily traders. There is a hidden, single differentiating aspect that bonds all those together, which is different horizon of time used for taking action. It is time horizon, not information, that generates also stability, with other words, liquidity in the markets.

There is a fundamental difference between the Efficient Market Hypothesis (EMH) and the Fractal Market Hypothesis (FMH) in terms of information embedded in trading prices: in the case of EMH, prices reflect all available information so they become unpredictable in relation to that amount of information, while FMH postulates that prices do not necessarily incorporate all available information, which generates a certain degree of predictability. On the other hand, according to EMH, prices reflect and incorporate instantly all available information (definiton of efficient market), so succesive price changes are attributable only to random process of generating new information (this is the true meaning of randomness, not information is random, but the process of penetration of new information is random). This means that the new information, and only it, can dictate future price changes, and as the process is random, future changes are random too. This means that predictibility is impossible, according to EMH. On the other hand, investors take decisions base on different time horizons, meaning that “significant information” for one investor may be categorised as “insignificant” to other, under different time horizons. This means that as new information appears into markets, some investors will not trade it, generating some degree of predictability.

3. Limitations of EMH. Liquidity Effect

Ball (2009) concludes the most evident limitations of the EMH, amongst: (i) information is an exogenous factor for the price function, (ii) information obtaining is cost-free, (iii) there is no liquidity effect over price. According to those limitations, FMH considers that, as respect to (i), price is an exogenous factor, but more important than price, we have different time horizons based on which different tipologies of investors take decisions. Based on this, the more heterogenous is the pool of investors, the more diversified will be the horizons they trade, the more different they will react or not to the same information, and they will value it quite in opposite manners. The diversity of tipology of investors will assure liquidity in the market. For (ii), obtaining information is free only for public sources. Even in such cases, the cost of processing and interpretation of that information is not free, and highly depends on the degree of sophistication of the investor’s profile. Also, this is available for normal periods. In case of “turbulent” periods, investors do not have the same ability to uniformly process information. Liquidity effect over price is also not considered under EMH (iii), but as liquidity decreases, and drops down, accordingly to an increasing trading volume, instability generates big price movements. Stability of the market is a matter of liquidity. And as expressed before in the paper, liquidity is available with many investors with different time horizons. For normal periods, liquidity generates market stability in this form. On the contrary, in periods when market loses its structure, all the investors have the same trading horizon. As a result, market becomes unstable, as liquidity fades away. This translates into losing the long-term investors, or their behaviour transformed into short term, meaning the whole market trades on the same information set, which may be technical or crowd phenomenon. For this, the market horizon becomes short term under uncertainty, as effect transforming the long-term information into unreliable or useless. Thus, largest crashes have occured when liquidity was low and trading volume is very high (trading volume means unbalanced supply and demand). EMH cannot explain price crashes, but when liquidity vanishes, getting the “fair” price does not seem so important as executing the trade at any cost. Thus, price is close to “fair” only correlated with liquidity, otherwise, investors are willing to take any price, whether „fair” or not.

4. Capital Markets: Non-Linear Dynamical Complex Systems

The classical approach regarding capital markets, based on the normal distribution assumption, is mathematically formulated using linear models. Despite this, the behaviour of participants into markets cannot be reduced using deductive linear models, but only for simplicity and conceptual elegance. The limitations of the econometric and statistical analysis are synthesized by Peters ():

1. Classical econometric analysis refers to the theory of equilibrium, in other words, if there are no external influences (external exogenous factors), the systems are in equilibrium, and all the forces of an economic system tend to balance. In the case the system is perturbed by external factors, the system tends to rebalance, this being its basic condition, the equilibrium state. However, examples in nature show that the system is far from equilibrum, and this is its natural state. However, free-market economies are evolving structures. Attempts to control them (state intervention) in order to make them more stable have failed due to the vital importance of far-from equilibrum conditions, so necessary for their development. More, free-capital markets are living structures, any intervention in their processes for stability may tend either to their collapse (death), or to un-natural functioning patterns. Attempt to model behavior of these systems using equilibrum theories generates disastrous results.

2. Classical mathematical theories refer to the time factor, in the sense that it approaches the concept that markets and the economy have no memory, or that their memory is very short. At best, econometrics uses short-term memory, but its effects disappear quickly, and the concept that a present event affects the future is completely foreign to current econometric science. EMH suggests that future price changes are generated by future information only, and not to present price value. As a consequence, past does not influence present, present does not influence future. But is it really the way markets function?

3. Also, the qualitative aspect involved in the human psychological decision-making process is completely neglected. Psychologically, people, as part of economic and stock market mechanisms, are influenced by what has happened in the past, and expectations about the future are influenced by recent experiences. This process is called, in behavioral finance, the feedback effect, in other words, the past influences the present and the present influences the future. “The rational investor” theory supposes that this rational person is not affected by past events, except maybe very recent ones. However, real feedback systems involve long-term trends and correlations, due to the fact that the memory of events from the more distant past can still affect the decisions made in the present. All these aspects lead to a so-called „disorder” or state of anomy of the markets, in which the classic, clear, simple, optimal solutions cannot be applied. Instead, several possible solutions are available.

These limitations generate profound bias in understanding markets as phenomenon. Attempt to assimilate markets as complex non-linear dynamical systems is far closer to their real identity. From this perspective, basic characteristics were derived:

feedback systems - past influences the present, and the present influences the future, in other words, P (t + 1) is a product of P (t).

existence of critical levels that allows more equilibrium points and stability bands.

self-similarity, feature characteristic for non-linear feedback processes. The complexity of such a system arises when the system is far from equilibrium.

the sensitive dependence on the initial conditions, in other words, P (t + n) is dependent, but still very different from P (t).

FMH allows understanding the behavior of market participants through the lens of complex dynamical non-linear processes. Benoit Mandelbrot considered time series of historical prices as deduced from "fractional brownian motions", and labeled them as "fractal series". Basic characteristic is correlation between different events, and between time intervals. For stying those time series, the whole history of the price must be accounted. Also, according to FMH, each observation „carries" a memory of all the events that precede it. This is not a short-term „Markovian" memory, but a different, long-term one that should be „eternal". The importance of recent events upon actual price is higher than those of more distant ones, of a longer period of time ago, but there is still residual influence. On a larger scale, a system that displays Hurst statistics is the result of a long series of interconnected events. Basically, what is happening today influences the future, and what happened some time ago influences what is happening today.

From this perspective, FMH studies and applies along time series with large period. Reffering to capital market, the entire historical price time series must be applied to. This implies both time factor as maximal, despite EMH, and phycological behavioral impact of very distant events still discounting into actual price, in trading strategies.

5. Mathematical Tool for Testing FMH – R/S Analysis

Simplified mathematical tool used for testing FMH was developed by Hurst, and taken over by numerous empirical studies, for enlarging its sphere of application into economics and capital markets, is described below:

For a given time series t, with „u" observations: (1)

where:

$X_{t,N}$ = cumulated deviation over N periods;

e_u = rate of return for period „u";

M_N = mean value of e_u over the N periods;

Range is equivalent to the difference between maximum and minimum levels attained for above equation:

$R = Max(X_{t,N}) - Min(X_{t,N})$, (2)

where:

R = range of X;

Max(X)= Maximum value for X;

Min(X) = Minimum value for X.

Furthermore, this interval should be divided to the stadard deviation of the original observations. This range rescaled is supposed to increase with time. The final relationship was formulated by Hurst:

$R/S = (a\cdot N)^H$ (3)

where:

R/S = Rescaled Range

N = number of observations

a = constant

H = Hurst Exponent.

For estimating the Hurst exponent, the above equation must be logarithmed as below:

$\log(R/S) = H\cdot\log(N) + \log(a)$ (4)

For a certain value of n, this procedure will generate a point on the graph log (R / S) vs log (n). Further, repetition of the method should be done for n + 10, n + 20, each time, generating another point on the graph, and completing up to n + 100. The slope of the graph is an estimate of the Hurst exponent.

Identifying the slope of the R / S log / log graph relative to N can give us a clue to identify the Hurst coefficient. However, estimating the Hurst coefficient does not provide any clues about the time series distribution.

Rescaled range analysis is still considered a robust tool for determining the memory effect, or the brownian character of a time series.

Taking into consideration values that can be obtained for the Hurst exponent, we can identify three categories:

(1) $H = 0.5$ - pure brownian motion

(2) $0 \leq H \leq 0.5$ – anti-persistent time series (probability of a different price change is 1-H)

(3) $0.5 < H < 1$ – persistent time series.

Persistent time series, defined by $0.5 < H < 1$, are fractal because they can be represented by fractional Brownian motions. Most of the economic time series, series of time generated by natural phenomena are persistent time series. As a conclusion, Hurst exponent determines the probability that if the last price change was positive, the next price would be positive, too. The closest to 1, the strongest the probability the next move would be identical to the prior.

6. Validity of the Hurst Exponent

Despite the fact that Hurst exponent may offer abnormal results, the question to ask is still if this estimation is valid or not. One explanation may consist in insufficient time series, or R/S analysis doesn't function the way it predicts. Except those cases, a particular value of H different of 0.5 can have only two possible explanations:

Therei is a long term memory component within the existing time series, meaning every observation is correlated in a manner with all the previous and future observations.

Analysis itself is biased, and, as a consequence, a biased value for Hurst doesn't imply a long-term memory effect.

Hurst exponent is a good indicator over the memory effect over price during time. R/S analysis is a robust statistical measure of independent processes. A value of H=0.5 does not imply a random Gaussian process. It only assumes there is no memory effect, with other words, any independent system, either Gaussian or not, may generate a value for H equal to 0.5.

6.1. R/S Analysis applied on the Romanian capital market. Case Study: Company Alro S.A.

Testing FMH on the Romanian Capital market was conducted by applying the R/S analysis on the historical price time series of company ALRO S.A. In order to test the fractalic character of the time series, all the price observations, since first day of listing at BVB, were taken into consideration.

The succesive equations above mentioned were implemented inside an informatic script, for the rapid implementation of the algoritm.

Nevertheless, one necessary condition for the implementation of the R/S analysis was a predetermination of the parameter n. After this input value established, the algorithm apllies for n+10, n+20, ..., m+100. Metescu (2015), in postdoctoral paper „Modelling stock market processes using FMH" observed the fact that the value of the Hurst exponent seemed to depend upon the value of the parameter „n", and also upon the value of the step parameter, from 10, 20, ..., 100 steps going further with the algorithm. There were performed multiple tests using two types of random walk, Gaussian and uniform, in the range (-1,1), using various combinations of values for n and for values attributed to step. Final results between the Hurst exponent, the total number of observations, the initial number of observations and the chosen step, illustrated for both „random walks", are illustrated in the tables below, Table A and Table B.

Table A. Estimations for the Hurst Exponent for a Gaussian Random Walk and Different Combinations for Parameters „n" and „step"

	1	2	3	4	5	10	15	20
200	0.6014	0.6018	0.6022	0.6026	0.6031	0.6052	0.6073	0.6095
400	0.6053	0.6058	0.6053	0.6068	0.6074	0.6099	0.6026	0.6150
600	0.6691	0.6696	0.6671	0.6705	0.6710	0.6734	0.6682	0.6781
800	0.9270	0.9271	0.9272	0.9272	0.9273	0.9277	0.9281	0.9284
1000	1.2584	1.2588	1.2566	1.2596	1.2600	1.2620	1.2379	1.2658
1200	1.6969	1.6974	1.6909	1.6984	1.6988	1.7011	1.6869	1.7048
1400	2.6047	2.6033	2.6018	2.6005	2.5989	2.5918	2.5851	2.5777
1600	4.6313	4.6249	4.6213	4.6122	4.6057	4.5746	4.5706	4.5146
1800	3.9078	3.9087	3.9226	3.9102	3.9109	3.9153	3.9516	3.9238

Table B. Estimations for the Hurst Exponent for an Uniform Random Walk and Different Combinations for Parameters „ n" and „step"

	1	2	3	4	5	10	15	20
200	0.8273	0.8276	0.8279	0.8282	0.8285	0.8301	0.8318	0.8331
400	0.7221	0.7226	0.7229	0.7238	0.7243	0.7271	0.7266	0.7318
600	0.5215	0.5222	0.5214	0.5236	0.5243	0.5277	0.5275	0.5345
800	0.4594	0.4600	0.4607	0.4614	0.4620	0.4653	0.4684	0.4714
1000	0.4749	0.4759	0.4750	0.4781	0.4791	0.4843	0.4695	0.4943
1200	0.5713	0.5720	0.5660	0.5733	0.5739	0.5772	0.5637	0.5838
1400	1.5324	1.5299	1.5273	1.5248	1.5223	1.5100	1.4983	1.4855
1600	2.9579	2.9547	2.9525	2.9486	2.9455	2.9302	2.9226	2.9025
1800	2.7890	2.7891	2.7950	2.7893	2.7893	2.7896	2.8057	2.7909

As results are displayed, the value for parameter „n" = 600, in both cases, represent the best approximation, close to 0.5 for the Hurst exponent, in the case parameter „step" = 10. From these estimations, we would extract the value of the parameter „n" = 600, and use this predetermined value for our algorithm for determining the values of the Hurst Exponent for the company ALRO S.A.

The results obtained for the Hurst Exponent are displayed in the following graphical illustrations (Graph 1, 2, 3), print-screens from FRACTAL – RISK platform, based on the script that implements the R/S analysis as below, and developed using a graphical user interface:

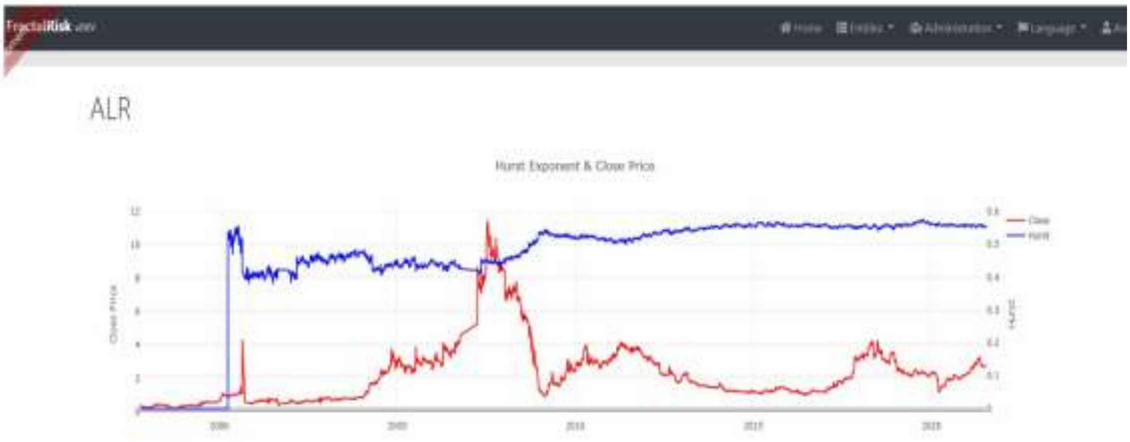

Figure 1. Print Screen from FRACTAL-RISK: Estimation of the Hurst Exponent for the Closing Daily Prices of ALRO S.A., for the Period since First Trading Day, 16/10/1997 until 16/07/2021 (Full Length of Time)

As noticed in the results displayed in the figure above, values for the Hurst exponent can be divided into two categories, corresponding to two time intervals: first, between 2000 and 2009, where H<0.5 except some exceptions, and values between 2009 and 2020, where H>0.5.

Explanation resides in the fact that for the first group, Hurst determines an anti-persistent trend, meaning tendency to reverse, and the process generating process may be described as a mean-reverting process or ergodic process. Basically, if the last price movement was in one direction, the probability that the next price change to be in the opposite direction is pretty high (1-H). Main characteristic is the lack of memory over price movements. The „strength" of the anti-persistent behavior of price depends on how close to zero the Hurst coefficient is. The closer the Hurst exponent is positioned to zero, the more volatile the trend, and the more frequent the changes. Basically, the value doesn't reach the neighborhood of zero, it is stabilised between 0.4 and 0.5. As a result, there is no stability, according to FMH, meaning no liquidity. One good explanation may be the under-development of the Romanian capital market, lack of experience and interest for potential and actual traders.

For resolution purpose, the estimations for the Hurst exponent, along with the closing daily prices are displayed in the figure below, corresponding the time interval 1997 – 2009 (zoom into data displayed):

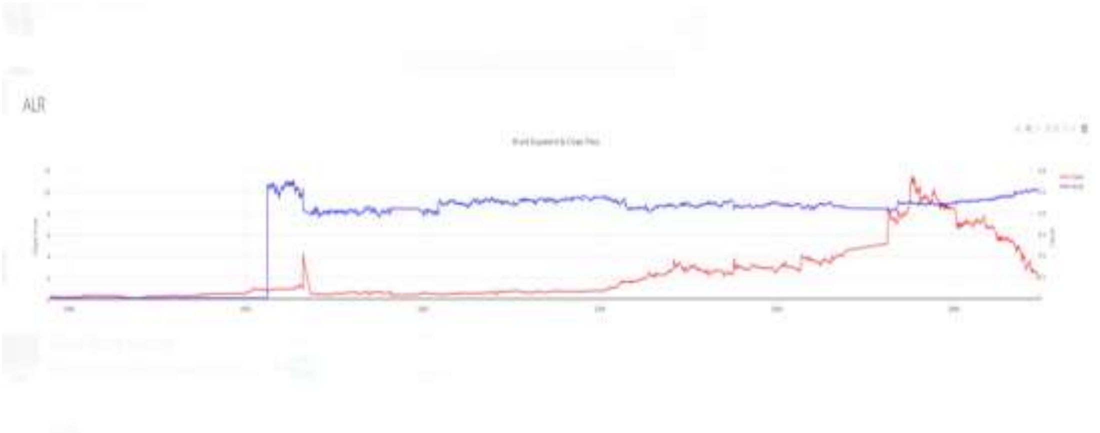

Figure 2. Print Screen from FRACTAL-RISK: Estimation of the Hurst Exponent for the Closing Daily Prices of ALRO S.A., for the Period Since First Trading day, 16/10/1997 Until 2009 (Zoom Function)

The second group of values for the Hurst exponent display values above 0.5, which would be equal to the random process. Hurst exponent determines a persistent trend, meaning that there may be some memory effect presented. Every price change is determined by a series of past movements, and every event is shaped by a serial interconnection of previous events. Overall, the memory effect is not very pronounced, as values do not exceed the 0.6 limit. The importance of distant past event is very small, as for recent events. Probability that the trend modifies is smaller (1-H), meaning the “strength” of the persistent behavior is above 0.5, but still below 0.6. As for the memory effect, there is some long-term memory effect presented in the market. This may be an indicator for increasing market stability, which translates into increasing liquidity, and a balanced trading volume. Causes for this phenomenon are multiple, among: the development of the Romanian capital market, increasing presence of more educated investors into the market, inflow of money by international funds investing in Romanian stocks. As postulated by FMH, the Hurst exponent simply quantifies the memory effect, in a market with increasing stabililty. As FMH suggests, the stability of the market indicates the presence of a heterogenous spectrum of investors, from daily traders and speculators to investment funds, generating a stable structure of the Romanian capital market. Also, the investment horizons of the partipants differ significantly, generating liquidity and financial health for the capital market. As for the closing prices, the values for the Hurst exponent suggest that they should be almost similar with their „fair values”, due to the stability generated in the market. As for the smoothness of the Hurst line, increased market maturity generates this flatness. There are no big price changes, nor extreme fluctuations, generated by an unbalanced trading volume, combined with small liquidity.

Again, the estimations for the Hurst exponent, along with the closing daily prices are displayed in the graph below, corresponding the time interval 2009 – 2021 (zoom into data displayed):

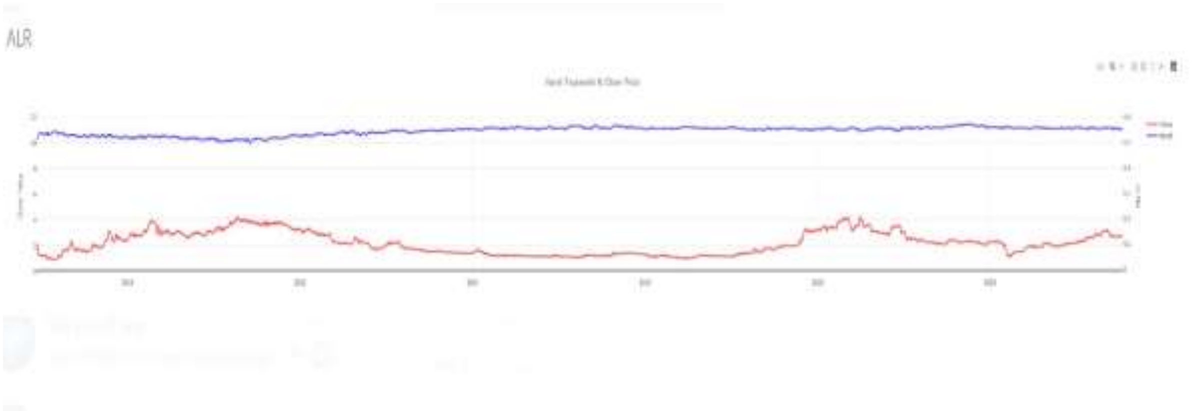

Figure 3. Print Screen from FRACTAL-RISK: Estimation of the Hurst exponent for the Closing Daily Prices of ALRO S.A., for the Period Since 2009 – 2021 (Zoom Function)

The particular case for H=0.5, when the first range of values for H (<0.5) reaches the value of H=0.5 is specific for a random process, meaning an independent system. In this case, as compared to the price movement, the Hurst exponent indicates precisely an abrupt decline in price, meaning a disproportion between liquidity and trading volume, where trades are executed at any price. Basically, short and long term investors trade under same horizons, due to uncertainty, which is very present in the market. Also, there is no memory effect, prices reflect only a deep fear emotion of the participants in the transactions.

7. Conclusions

Romanian capital market evolved as from an unsecured investment environment to a more stable position, as investment opportunity at international scale. R/S Analysis is a robust indicator, for determining even transition from uncertaity to stability. The evolution of historical daily closing prices of ALRO S.A., starting as anti-persistent process, passing through random walk phase, and reaching to persistent processes is the evolution from a very low stability market to a more stable one. The Hurst exponent precisely offers information about maturity, stability and memory effect inside markets.

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