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The Determination of a Company Production under the Conditions of Minimizing the Production Costs, but Also Profit Maximization

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Abstract. The paper deals with the problem of determining the production of a company under the conditions in which it wants both the minimization of the production costs and the maximization of the profit.

Keywords: production function; Cobb-Douglas; profit

1. Introduction

Let us consider a firm F whose activity is formalized using a production function Q which depends on a number of production factors $x_1,...,x_n$, $n\geq 2$. In order to ensure its competitiveness on the market, its main purpose is to reduce its total cost which will implicitly lead to the output of its products at the lowest possible cost. On the other hand, the company wants to maximize its profit. For example, we will consider the production function as Cobb-Douglas type, which is equivalent to a constancy of the elasticities of production in relation to the factors of production, which is not restrictive, at least for a limited time.

The Cobb-Douglas function has the following expression:

$$Q:D\subset \mathbf{R}_+^n-\{0\}\to\mathbf{R}_+,\ (x_1,...,x_n)\to Q(x_1,...,x_n)=Ax_1^{\alpha_1}...x_n^{\alpha_n}\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in D,\ A\in\mathbf{R}_+^*,\ \alpha_1,...,\alpha_n\in\mathbf{R}_+^*.$$

$$Q'_{x_i}=A\alpha_i x_1^{\alpha_1}...x_i^{\alpha_i-1}...x_n^{\alpha_n}=\frac{\alpha_i Q}{x_i},i=\overline{1,n}$$

The main indicators are:

- $\eta_{x_i}=\frac{\partial Q}{\partial x_i}=A\alpha_i x_1^{\alpha_1}...x_i^{\alpha_i-1}...x_n^{\alpha_n}=\frac{\alpha_i Q}{x_i},\ i=\overline{1,n};$
- $w_{x_i}=\frac{Q}{x_i}=Ax_1^{\alpha_1}...x_i^{\alpha_i-1}...x_n^{\alpha_n}=\frac{Q}{x_i},\ i=\overline{1,n};$
- $RMS(i,j)=\frac{\alpha_i x_j}{\alpha_j x_i},\ i,j=\overline{1,n};$
- $\varepsilon_{x_i}=\frac{\eta_{x_i}}{w_{x_i}}=\alpha_i,\ i=\overline{1,n};$
- $\sigma_{ij}=-1,\ i,j=\overline{1,n}.$

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2. The costs of the Cobb-Douglas production function

Considering now the problem of minimizing costs for a given production Q_0, where the prices of inputs are p_i, $i=\overline{1,n}$, we have:

$$\begin{cases} \min \sum_{k=1}^{n} p_k x_k \\ A x_1^{\alpha_1} \ldots x_n^{\alpha_n} \geq Q_0 \\ x_1, \ldots, x_n \geq 0 \end{cases}$$

From the obvious relations: $\begin{cases} \frac{\alpha_1}{p_1 x_1} = \ldots = \frac{\alpha_n}{p_n x_n} \\ A x_1^{\alpha_1} \ldots x_n^{\alpha_n} = Q_0 \end{cases}$ we obtain: $\begin{cases} x_k = \frac{\alpha_k p_n}{\alpha_n p_k} x_n, k = \overline{1,n\text{-}1} \\ A x_1^{\alpha_1} \ldots x_n^{\alpha_n} = Q_0 \end{cases}$ and from the second equation: $A \frac{p_n^{\sum_{k=1}^{n-1} \alpha_k} \prod_{k=1}^{n-1} \alpha_k^{\alpha_k}}{\alpha_n^{\sum_{k=1}^{n-1} \alpha_k} \prod_{k=1}^{n-1} p_k^{\alpha_k}} x_n^{\sum_{k=1}^{n} \alpha_k} = Q_0$. Noting $r=\sum_{k=1}^{n} \alpha_k > 0$, we finally obtain:

$$\bar{x}_k = \frac{(\prod_{k=1}^{n} p_k^{\alpha_k})^{1/r}}{(\prod_{k=1}^{n} \alpha_k^{\alpha_k})^{1/r}} \frac{\alpha_k}{p_k} \frac{Q_0^{1/r}}{A^{1/r}}, \ k=\overline{1,n}$$

The total cost is:

$$TC(Q_0)=\sum_{k=1}^{n} p_k \bar{x}_k = \frac{(\prod_{i=1}^{n} p_i^{\alpha_i})^{1/r}}{(\prod_{i=1}^{n} \alpha_i^{\alpha_i})^{1/r}} \frac{r Q_0^{1/r}}{A^{1/r}}.$$

The marginal cost is:

$$TC_m(Q_0)=\sum_{k=1}^{n} p_k \bar{x}_k = \frac{(\prod_{i=1}^{n} p_i^{\alpha_i})^{1/r}}{(\prod_{i=1}^{n} \alpha_i^{\alpha_i})^{1/r}} \frac{Q_0^{(1-r)/r}}{A^{1/r}}.$$

Noting, for simplicity: $\gamma = \frac{(\prod_{i=1}^{n} p_i^{\alpha_i})^{1/r}}{(\prod_{i=1}^{n} \alpha_i^{\alpha_i})^{1/r}} \frac{1}{A^{1/r}} > 0$ and $s = \frac{1-r}{r}$, follows: $TC_m(Q_0)=\gamma Q_0^s$, $TC(Q)=r\gamma Q_0^{\frac{1}{r}} = \frac{\gamma}{s+1} Q_0^{s+1}$ because $sr=1-r \Leftrightarrow r(s+1)=1 \Leftrightarrow r=\frac{1}{s+1}$.

Let’s note that, because $r>0$, we have $s \in (-1,\infty)$.

Consider the profit of the company: $\pi(Q)=p(Q)Q-TC(Q)$

If $p(Q)=a-bQ$, $a,b>0$, we have: $\pi(Q)=aQ-bQ^2-TC(Q)=aQ-bQ^2-\frac{\gamma}{s+1}Q^{s+1}$.

hence, the extremely necessary condition of profit becomes:

$\pi'(Q)=a-2bQ-TC_m(Q)=0$

otherwise: $a-2bQ-\gamma Q^s=0$.

Also, the necessary and sufficient condition for maximization is:

$\pi''(Q)=-2b-TC_m(Q)'<0$

that is:

$-2b-\gamma s Q^{s-1}<0 \Leftrightarrow Q^{s-1}>-\frac{2b}{\gamma s}$ if $s>0$ which is obvious because $Q>0$ and $Q^{s-1}<-\frac{2b}{\gamma s}$ if $s \in (-1,0)$.

If $r=1$, namely $s=0$, then: $a-\gamma-2bQ=0 \Rightarrow Q=\frac{a-\gamma}{2b}$.

In this case, the maximization condition returns to: $-2b<0$ which is true.

If $r\neq1$, namely $s\neq0$, results:

$a-2bQ-\gamma Q^s=0$, $s\in(-1,0)\cup(0,\infty)$.

Let the functions $\pi':(0,\infty)\to\mathbf{R}$, $\pi'(Q)=a-2bQ-\gamma Q^s$ and $\pi''(Q)=-2b-\gamma sQ^{s-1}$, $\pi'''(Q)=-\gamma s(s-1)Q^{s-2}$.

Case 1: $s>0$

As in this case, $\pi''(Q)<0$ we have that π' it is strictly decreasing. Because $\lim_{Q\to0}\pi'(Q)=a>0$, $\lim_{Q\to\infty}\pi'(Q)=-\infty$ it turns out that the equation $\pi'(Q)=0$ it has only one strictly positive root $Q_1\in(0,\infty)$ which, by virtue of the above, is a local maximum point.

Case 2: $s<0$

In this case, $\pi'''(Q)<0$ therefore π'' is strictly decreasing. But we have: $\lim_{Q\to0}\pi''(Q)=\infty$, $\lim_{Q\to\infty}\pi''(Q)=-2b<0$ so the equation $\pi''(Q)=0$ has a single positive root Q^* that satisfies the relationship: $-2b-\gamma sQ^{*s-1}=0$ or otherwise: $Q^*=\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}$. Thus, $\pi''(Q)>0$ $\forall Q\in(0,Q^*)$ and $\pi''(Q)<0$ $\forall Q\in(Q^*,\infty)$, namely π' is strictly increasing on $(0,Q^*)$ and strictly decreasing on (Q^*,∞).

Because: $\lim_{Q\to0}\pi'(Q)=-\infty$, $\pi'(Q^*)=a-2bQ^*-\gamma Q^{*s}=a-2b\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}-\gamma\left(-\frac{2b}{\gamma s}\right)^{\frac{s}{s-1}}=$ $\left(a+\frac{2b}{s}\right)-(2b+\gamma)\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}$, $\lim_{Q\to\infty}\pi'(Q)=-\infty$ we have:

Case 2.1

If $\left(a+\frac{2b}{s}\right)-(2b+\gamma)\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}<0$ then the equation $\pi'(Q)=0$ has no positive roots. In this case, π has constant monotony. How $s\in(-1,0)\Rightarrow\lim_{Q\to0}\pi(Q)=0$, $\lim_{Q\to\infty}\pi(Q)=-\infty$ so the profit being negative, the company is at a loss and therefore the only option is to stop production.

Case 2.2

If $\left(a+\frac{2b}{s}\right)-(2b+\gamma)\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}=0$ then the equation $\pi'(Q)=0$ has the root $Q^*=\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}$.

But $\pi''(Q^*)=0$ and $\pi'''(Q)=-\gamma s(s-1)Q^{*s-2}<0$ so π has no extreme point. On the other hand, in this case $\pi'(Q)\leq0$ so π is decreasing. In this case, as production increases, profit will decrease. The maximum profit will therefore be recorded for $Q = 0$, meaning the company will not produce.

Case 2.3

If $\left(a+\frac{2b}{s}\right)-(2b+\gamma)\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}>0$ then the equation $\pi'(Q)=0$ has two positive roots: $Q_1\in(0,Q^*)$. $Q_2\in(Q^*,\infty)$. How $\pi''(Q_1)>0$ follows that Q_1 is a local minimum point, and how $\pi''(Q_2)<0$ it turns out that Q_2 is a local maximum point.

So let the equation: $0=\pi'(Q)=a-2bQ-\gamma Q^s$ with the solution $Q^{**}>Q^*$. Thus: $a-2bQ^{**}-\gamma Q^{**s}=0$ or otherwise: $Q^{**s}=\frac{a-2bQ^{**}}{\gamma}$.

We have $\pi(Q^{**})$=$aQ^{**} - bQ^{**2} - \frac{\gamma}{s+1}Q^{**(s+1)}$= $aQ^{**} - bQ^{**2} - \frac{\gamma}{s+1}\frac{a-2bQ^{**}}{\gamma}Q^{**}$=

$aQ^{**} - bQ^{**2} - \frac{aQ^{**}-2bQ^{**2}}{s+1}$= $\frac{asQ^{**}+b(1-s)Q^{**2}}{s+1}$>0.

Therefore, for production Q^{**} which satisfies the equation: $a - 2bQ^{**} - \gamma Q^{**s} = 0$ the company will record a maximum profit.

3. Partial Conclusions

- r=1 ⇔ s=0 implies Q=$\frac{a-\gamma}{2b}$;
- r<1 ⇔ s>0 implies that Q is the root of the equation $a - 2bQ - \gamma Q^{s} = 0$;
- If $\left(a+\frac{2b}{s}\right)$-(2b+γ)$\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}$≤0, s=$\frac{1-r}{r}$ then the company ceases its activity;
- If $\left(a+\frac{2b}{s}\right)$-(2b+γ)$\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}$>0, s=$\frac{1-r}{r}$, s∈(-1,0) implies that Q is the root of the equation $a - 2bQ - \gamma Q^{s} = 0$ which additionally satisfies the condition Q>$\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}$.

4. The Solution of the Nonlinear Equation

Let the equation: a-2bQ-γQ^{s}=0, s∈(-1,∞), Q>0 and f:(0,∞)→**R,** f(Q)=a-2bQ-γQ^{s}, f’(Q)=-2b-γsQ^{s-1}, f’’(Q)=-γs(s-1)Q^{s-2}.

For convergence, the function must have the same monotony and concavity over the interval (a, b) in which the root is found. The starting point is the one for which f(Q_0)f’’(Q_0)>0.

Case 1: s>1

How f’’ (Q)=-γs(s-1)Q^{s-2}<0 it turns out that Q_0 is chosen so that f(Q_0)<0. On the other hand, it turns out that f’ is strictly decreasing. How f’ (0)=-2b<0 it follows that f is strictly decreasing. But $\lim_{Q\to\infty} f(Q)$=-∞ implies that we will choose Q_0 large enough. How $f\left(\frac{2as}{2b(s-1)}\right)$= $a - 2b\frac{2as}{2b(s-1)} - \gamma\left(\frac{as}{2b(s-1)}\right)^{s}$ =$-a\ \frac{s+1}{s-1} - \gamma\left(\frac{as}{2b(s-1)}\right)^{s}$ <0 we have that Q_0=$\frac{2as}{2b(s-1)}$.

Case 2: s=1

The equation becomes a-(2b+γ)Q=0 from where Q=$\frac{a}{2b+\gamma}$.

Case 3: s∈(0,1)

How f’’(Q)=-γs(s-1)Q^{s-2}>0 it turns out that Q_0 is chosen so that f(Q_0)>0. In this case, f’ is strictly increasing and how $\lim_{Q\to 0} f'(Q)$=-∞, $\lim_{Q\to\infty} f'(Q)$=-2b<0 it turns out that f’ is strictly negative, so f is strictly decreasing. How f(0)=a>0 follows that Q_0=0.

Case 4: s∈(-1,0)

How $f''(Q)=-\gamma s(s-1)Q^{s-2}<0$ it turns out that Q_0 is chosen so that $f(Q_0)<0$. is chosen so that, f' is strictly decreasing and how $\lim_{Q\to 0} f'(Q)=-\infty$, $\lim_{Q\to\infty} f'(Q)=-2b<0$ it turns out that f' is strictly negative, so f is strictly decreasing. How $\lim_{Q\to\infty} f(Q)=-\infty$ implies that we will choose Q_0 large enough. On the other hand, $Q_0>Q^*$ that is $Q_0>\left(-\frac{2b}{\gamma s}\right)^{\frac{1}{s-1}}$.

Applying Newton's recurrence formula, it results:

$$Q_{(n+1)} = Q_{(n)} - \frac{f\left(Q_{(n)}\right)}{f'\left(Q_{(n)}\right)} = Q_{(n)} - \frac{a - 2bQ_{(n)} - \gamma Q_{(n)}^{s}}{-2b - \gamma s Q_{(n)}^{s-1}} = \frac{\gamma(1-s)Q_{(n)}^{s} - a}{-2b - \gamma s Q_{(n)}^{s-1}}$$

Therefore: $Q_{(n+1)} = \frac{\gamma(1-s)Q_{(n)}^{s}-a}{-2b-\gamma s Q_{(n)}^{s-1}}$, $n\geq 0$.

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