THE 16TH EDITION OF THE INTERNATIONAL CONFERENCE
EUROPEAN INTEGRATION
REALITIES AND PERSPECTIVES

Performance and Risks in the European Economy

Behavior of Production Factors when Changing their Price for a Cobb-Douglas Production Function

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Abstract: The paper deals with the behavior of the factors of production when changing their price as well as the total cost for a Cobb-Douglas production function.

Keywords: production function; Cobb-Douglas; total cost

1. Introduction

Let us consider a firm F whose activity is formalized using a production function Q which depends on a number of production factors $x_1,...,x_n$, $n\geq 2$. In order to ensure its competitiveness on the market, its main purpose is to reduce its total cost which will implicitly lead to the output of its products at the lowest possible cost. On the other hand, the company wants to maximize its profit. For example, we will consider the production function as Cobb-Douglas type, which is equivalent to a constancy of the elasticities of production in relation to the factors of production, which is not restrictive, at least for a limited time.

The Cobb-Douglas function has the following expression:

$Q:D\subset\mathbf{R}^n_+-\{0\}\to\mathbf{R}_+$, $(x_1,...,x_n)\to Q(x_1,...,x_n)=\gamma x_1^{\alpha_1}...x_n^{\alpha_n}\in\mathbf{R}_+$ $\forall(x_1,...,x_n)\in D$, $\gamma\in\mathbf{R}^*_+$, $\alpha_1,...,\alpha_n\in\mathbf{R}^*_+$.

$Q'_{x_i}=\gamma\alpha_i x_1^{\alpha_1}...x_i^{\alpha_i-1}...x_n^{\alpha_n}=\frac{\alpha_i Q}{x_i}$, $i=\overline{1,n}$

The main indicators are:

- $\eta_{x_i}=\frac{\partial Q}{\partial x_i}=\gamma\alpha_i x_1^{\alpha_1}...x_i^{\alpha_i-1}...x_n^{\alpha_n}=\frac{\alpha_i Q}{x_i}$, $i=\overline{1,n}$
- $w_{x_i}=\frac{Q}{x_i}=\gamma x_1^{\alpha_1}...x_i^{\alpha_i-1}...x_n^{\alpha_n}=\frac{Q}{x_i}$, $i=\overline{1,n}$
- $RMS(i,j)=\frac{\alpha_i x_j}{\alpha_j x_i}$, $i,j=\overline{1,n}$
- $\varepsilon_{x_i}=\frac{\eta_{x_i}}{w_{x_i}}=\alpha_i$, $i=\overline{1,n}$
- $\sigma_{ij}=-1$, $i,j=\overline{1,n}$

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2. Behavior of Production Factors when Changing their Price for a Cobb-Douglas Production Function

Considering now the problem of minimizing costs for a given production Q_0, where the prices of inputs are p_i, $i=\overline{1,n}$, we have:

$$\begin{cases} \min \sum_{k=1}^{n} p_k x_k \\ \gamma x_1^{\alpha_1} \dots x_n^{\alpha_n} \ge Q_0 \\ x_1, \dots, x_n \ge 0 \end{cases}$$

From the obvious relations: $\begin{cases} \frac{\alpha_1}{p_1 x_1} = \dots = \frac{\alpha_n}{p_n x_n} \\ \gamma x_1^{\alpha_1} \dots x_n^{\alpha_n} = Q_0 \end{cases}$ we obtain: $\begin{cases} x_k = \frac{\alpha_k p_n}{\alpha_n p_k} x_n, k = \overline{1,n\text{-}1} \\ \gamma x_1^{\alpha_1} \dots x_n^{\alpha_n} = Q_0 \end{cases}$ and from the second equation: $\gamma \frac{p_n^{\sum_{k=1}^{n-1}\alpha_k} \prod_{k=1}^{n-1} \alpha_k^{\alpha_k}}{\alpha_n^{\sum_{k=1}^{n-1}\alpha_k} \prod_{k=1}^{n-1} p_k^{\alpha_k}} x_n^{\sum_{k=1}^{n}\alpha_k} = Q_0$. Noting $r=\sum_{k=1}^{n}\alpha_k>0$, we finally obtain:

$\bar{x}_k=\frac{(\prod_{k=1}^{n} p_k^{\alpha_k})^{1/r}}{(\prod_{k=1}^{n} \alpha_k^{\alpha_k})^{1/r}} \frac{\alpha_k}{p_k} \frac{Q_0^{1/r}}{\gamma^{1/r}}$, $k=\overline{1,n}$

The total cost is:

$TC(Q_0)=\sum_{k=1}^{n} p_k \bar{x}_k=\frac{(\prod_{i=1}^{n} p_i^{\alpha_i})^{1/r}}{(\prod_{i=1}^{n} \alpha_i^{\alpha_i})^{1/r}} \frac{r Q_0^{1/r}}{\gamma^{1/r}}$.

Let us now calculate the influence of the price increase of one of the factors of production on the absolute value of that factor.

$\frac{\partial \bar{x}_k}{\partial p_i}=\frac{\alpha_k Q_0^{1/r} (\prod_{s=1, k\neq i}^{n} p_s^{\alpha_s})^{1/r}}{(\gamma \prod_{s=1}^{n} \alpha_s^{\alpha_s})^{1/r}} \frac{\frac{\alpha_i}{r} p_i^{\frac{\alpha_i}{r}-1} p_k - p_i^{\frac{\alpha_i}{r}} \delta_{ik}}{p_k^2}=\frac{\alpha_k Q_0^{1/r} (\prod_{s=1}^{n} p_s^{\alpha_s})^{1/r}}{(\gamma \prod_{s=1}^{n} \alpha_s^{\alpha_s})^{1/r}} \frac{\alpha_i p_k - r\delta_{ik} p_i}{r p_i p_k^2}$, $k,i=\overline{1,n}$

In particular:

$\frac{\partial \bar{x}_k}{\partial p_k}=-\frac{\alpha_k Q_0^{1/r} (\prod_{s=1}^{n} p_s^{\alpha_s})^{1/r}}{(\gamma \prod_{s=1}^{n} \alpha_s^{\alpha_s})^{1/r}} \frac{r-\alpha_k}{r p_k^2}$, $k=\overline{1,n}$

$\frac{\partial \bar{x}_k}{\partial p_i}=\frac{\alpha_k Q_0^{1/r} (\prod_{s=1}^{n} p_s^{\alpha_s})^{1/r}}{(\gamma \prod_{s=1}^{n} \alpha_s^{\alpha_s})^{1/r}} \frac{\alpha_i}{r p_i p_k}$, $k,i=\overline{1,n}$, $i\neq k$

The first differential of $\bar{x}_k$ is:

$$d\bar{x}_k = \sum_{i=1}^{n} \frac{\partial \bar{x}_k}{\partial p_i} dp_i=\frac{\alpha_k Q_0^{1/r} (\prod_{s=1}^{n} p_s^{\alpha_s})^{1/r}}{r p_k (\gamma \prod_{s=1}^{n} \alpha_s^{\alpha_s})^{1/r}} \left(-\frac{r-\alpha_k}{p_k} dp_k + \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{\alpha_i}{p_i} dp_i \right)=$$

$$\frac{\alpha_k Q_0^{1/r} (\prod_{s=1}^{n} p_s^{\alpha_s})^{1/r}}{r p_k (\gamma \prod_{s=1}^{n} \alpha_s^{\alpha_s})^{1/r}} \left(-\frac{r}{p_k} dp_k + \sum_{i=1}^{n} \frac{\alpha_i}{p_i} dp_i \right)$$

The factor $\bar{x}_k$ will decrease if and only if: $d\bar{x}_k<0 \Leftrightarrow -\frac{r}{p_k} dp_k + \sum_{i=1}^{n} \frac{\alpha_i}{p_i} dp_i<0 \Leftrightarrow \frac{r}{p_k} dp_k>\sum_{i=1}^{n} \frac{\alpha_i}{p_i} dp_i$ $\Leftrightarrow \frac{dp_k}{p_k}>\frac{\sum_{i=1}^{n} \frac{\alpha_i}{p_i} dp_i}{r}=\frac{\sum_{i=1}^{n} \frac{\alpha_i}{p_i} dp_i}{\sum_{i=1}^{n} \alpha_i}$. If we note: $\varphi_i=\frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i}\in(0,1)$ we obtain: $\frac{dp_k}{p_k}>\sum_{i=1}^{n} \varphi_i \frac{dp_i}{p_i}$ and $\sum_{i=1}^{n} \varphi_i=1$.

$\frac{dp_k}{p_k}>\frac{1}{1-\varphi_k}\sum_{\substack{i=1\\i\neq k}}^{n}\varphi_i\frac{dp_i}{p_i}$ or $\frac{dp_k}{p_k}>\frac{\sum_{\substack{i=1\\i\neq k}}^{n}\varphi_i\frac{dp_i}{p_i}}{\sum_{\substack{i=1\\i\neq k}}^{n}\varphi_i}$. If we note again: $\lambda_i=\frac{\varphi_i}{\sum_{\substack{i=1\\i\neq k}}^{n}\varphi_i}$, $i=\overline{1,n}$, $i\neq k$ we have: $\sum_{\substack{i=1\\i\neq k}}^{n}\lambda_i=1$.

The condition becomes: $\frac{dp_k}{p_k}>\sum_{\substack{i=1\\i\neq k}}^{n}\lambda_i\frac{dp_i}{p_i}$ On the other hand, the expression $\sum_{\substack{i=1\\i\neq k}}^{n}\lambda_i\frac{dp_i}{p_i}$ represents the convex combination of the relative variations of the factors except for p_k.

Therefore, a relative variation of the pk factor outside the convex combination (for clarification, the convex combination of two quantities is given by the inside of the segment, three sizes by the inside of the triangle etc.) will lead to a decrease of its absolute value.

For example, if $\frac{dp_i}{p_i}=C$, $C\in\mathbf{R}$ we will have that: $\frac{dp_k}{p_k}>C$ will lead to a decrease of absolute value of $\bar{x}_k$.

A concrete example is that of the labor force. In the conditions of inflationary pressures that increase the prices of other factors of production (electricity, liquid fuels etc.) by a percentage of C%, in order to maintain the existing labor force, wages must increase by at most C%.

More:

$$\frac{d\bar{x}_k}{\bar{x}_k}=-\left(\sum_{\substack{i=1\\i\neq k}}^{n}\alpha_i\right)\frac{dp_k}{p_k}+\sum_{\substack{i=1\\i\neq k}}^{n}\alpha_i\frac{dp_i}{p_i}=r\left(-(1-\varphi_k)\frac{dp_k}{p_k}+\sum_{\substack{i=1\\i\neq k}}^{n}\varphi_i\frac{dp_i}{p_i}\right)$$

which expresses the relative variation of the factor x_k as a function of the relative variations of the prices p_i.

Let us now calculate the influence of the price increase of one of the factors of production on the total cost of production.

$$\frac{\partial TC(Q_0)}{\partial p_i}=\frac{rQ_0^{1/r}\left(\prod_{\substack{k=1\\k\neq i}}^{n}p_k^{\alpha_k}\right)^{1/r}}{\left(\gamma\prod_{k=1}^{n}\alpha_k^{\alpha_k}\right)^{1/r}}\frac{\alpha_i}{r}p_i^{\frac{\alpha_i}{r}-1}=\frac{rQ_0^{1/r}\left(\prod_{k=1}^{n}p_k^{\alpha_k}\right)^{1/r}}{\left(\gamma\prod_{k=1}^{n}\alpha_k^{\alpha_k}\right)^{1/r}}\frac{\alpha_i}{rp_i}=TC(Q_0)\frac{\alpha_i}{rp_i}$$

Therefore:

$d\,TC(Q_0)=\sum_{i=1}^{n}\frac{\partial TC(Q_0)}{\partial p_i}dp_i=\sum_{i=1}^{n}TC(Q_0)\frac{\alpha_i}{rp_i}dp_i$ from where:

$$\frac{d\,TC(Q_0)}{TC(Q_0)}=\sum_{i=1}^{n}\frac{\alpha_i}{r}\frac{dp_i}{p_i}=\sum_{i=1}^{n}\varphi_i\frac{dp_i}{p_i}.$$

As a result of these calculations, the relative variation of the total cost will be found in the convex combination of the relative variations of the factors of production.

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